18.085 Exam 2 Professor Strang October 30, 2014

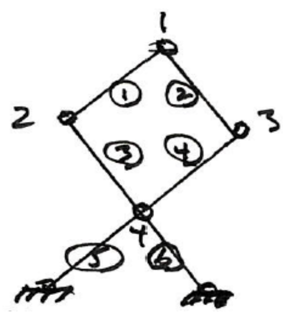
Grading

1
2
3
4

Your PRINTED name is: Solutions

Your class number is: ∞

1. (25 points)



(a) What is the first row of the matrix A for this truss, to give the stretching e_1 in bar 1 from the node displacements?

(b) Describe with pictures a full set of independent solutions to $\mathbf{Au} = \mathbf{0}$.

(c) What are the components $u_1^H, u_1^V, \ldots, u_4^H, u_4^V$ of *one solution* to $Au = 0$?

Solutions:

(a) The first row of A is $\begin{pmatrix} \cos\theta & \sin\theta & -\cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \end{pmatrix}$, where θ is the angle of bar 1.

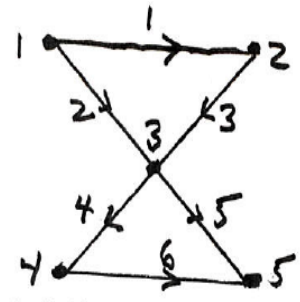
(b) Notice that A is of rank 6, thus there are $8 - 6 = 2$ independent solutions, and clearly both would be mechanisms. One mechanism is the rotation about node 4. The second mechanism is a "collapsing" mechanism, where the square part of the truss collapses outwards (like opening a book).

(c) The collapsing mechanism is: $u = \begin{pmatrix} 0 & -1/\sin\theta & -\cos\theta & -\sin\theta & \sin\theta & -\cos\theta & 0 & 0 \end{pmatrix}$.

The rotation mechanism is: $u = \begin{pmatrix} 1/\cos\theta & 0 & \cos\theta & \sin\theta & \sin\theta & -\cos\theta & 0 & 0 \end{pmatrix}$.

Either answer was accepted.

2. (25 points)



(a) Write down the incidence matrix A and find a complete set of independent solutions to $\mathbf{Au} = \mathbf{0}$. *What is the rank of* A?

(b) Find a complete set of independent solutions to $\mathbf{A^T w} = \mathbf{0}$ (Kirchhoff's Current Law), where $w = (w_1, \ldots, w_6)$ gives currents along the edges.

(c) Find a complete set of independent solutions to $\mathbf{A^T A u} = \mathbf{0}$. You could answer without multiplying $A^T A$ if you remember the inner product with u^T in Exam 1.

Solutions:

(a) There are 6 edges and 5 nodes, which means the incidence matrix A has dimensions 6×5. The rank of A is $r = n - 1 = 5 - 1 = 4$.

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

The set of independent solutions is just the line: $u = C \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ for $C \in \mathbb{R}$.

(b) The complete set of independent solutions to $A^T w = 0$ is governed by the number of closed loops, of which there are two (loop around top triangle, loop around bottom triangle). They are: $w_1 = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \end{pmatrix}^T$ and $w_2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 1 & -1 \end{pmatrix}^T$. These loops are a basis for the nullspace of A^T, and to check we have the correct number, we can verify that $5 - 6 + 2 = 1 =$ (# nodes) $-$ (# edges) $+$ (# loops).

(c) Notice that $A^T A$ has the same nullspace as A, since both satisfy $A^T A u = 0$ and $Au = 0$. Hence, this again is just the line $u = C \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}^T$ for $C \in \mathbb{R}$. There is no need to compute $A^T A$.

3. (25 points)

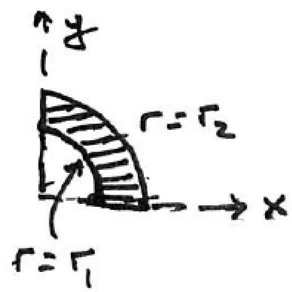
(a) Suppose $u(x, y)$ and $s(x, y)$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial s}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial s}{\partial x}$$

Take derivatives to show that BOTH u and s solve Laplace's equation.

(b) Show that the gradient of $\log r = \log \sqrt{x^2 + y^2}$ is a vector of length $\frac{1}{r}$ pointing in which direction?

(c) Using (b) find the flux from the line integral around the boundary of this piece of a ring $r_1 \leq r \leq r_2, 0 \leq \theta \leq \frac{\pi}{2}$. (You could check by thinking about div (grad u) in the double integral inside the ring).



Solutions:

(a) Laplace's Equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since u and s satisfy the Cauchy-Riemann equations, we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial s}{\partial x \partial y}$$

and

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\partial s}{\partial x}\right) = -\frac{\partial s}{\partial x \partial y}$$

The sum of the above two terms is 0, which means u and s satisfy Laplace's equation.

(b) $\nabla \log r = \nabla \log \sqrt{x^2 + y^2} = \left(\frac{1}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}}, \frac{1}{\sqrt{x^2+y^2}} \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{1}{r^2}(x, y).$

This vector has length $\dfrac{1}{r}$ since (x, y) is of magnitude r. The gradient points radially outwards.

(c) Notice that $\nabla \cdot \nabla \log r = 0$, and hence by the 2D Divergence Theorem, Flux $= 0$ away from the origin. Since the region does not touch the origin, again the flux through the region is zero.

4. (25 points) This question asks for the 4 by 4 element matrix K_2 and 4 by 1 element load vector F_2, using 4 hat functions $\phi_i = V_i$ to solve

$$-3u'' = \delta(x-0.3) \text{ with fixed } u(0) = u(1) = 0.$$

The entries of K and F are

$$K_{ij} = \int_0^1 3\,\frac{d\phi_i}{dx}\,\frac{d\phi_j}{dx}dx \qquad F_i = \int_0^1 \delta(x-0.3)\,\phi_i(x)\,dx$$

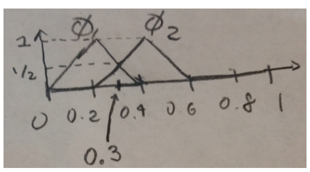
The matrix K_2 and the vector F_2 come from *integrating from* $h = 0.2$ *to* $2h = 0.4$. The trial/test functions are $\phi_i = 1$ at meshpoint $x = ih$ and $\phi_i = 0$ at other meshpoints. Notice: the spike at 0.3 is *halfway between* h *and* $2h$.

(a) Graph ϕ_1 and ϕ_2.

(b) Find the 4 by 4 element matrix K_2.

(c) Find the 4 by 1 element matrix F_2.

<u>Solutions:</u>

(a) 

(b) Notice that from our boundary $h = 0.2$ to $h = 0.4$, $K_{ij} = 0$ and $K_{ji} = 0$ for $i \in \{1,2\}, j \in \{3,4\}$ so it remains to find K_{11}, K_{12}, K_{21}, and K_{22}.

The upward and downward slopes are 5 and -5 respectively. So,

$$K_{11} = \int_{0.2}^{0.4} 3 \cdot (-5) \cdot (-5) dx = 75(0.2) = 15$$

Similarly, $K_{22} = 15$. By definition $K_{12} = K_{21}$, and in our boundary the two test functions ϕ_1 and ϕ_2 have opposite slope, meaning $K_{12} = K_{21} = -15$. The matrix is:

$$K_2 = \begin{pmatrix} 15 & -15 & 0 & 0 \\ -15 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(c) Here, notice $F_{13} = F_{14} = 0$, since the test functions ϕ_3 and ϕ_4 are 0 in the boundary. So, it remains to find F_{11} and F_{12}. Using the definition of the delta function,

$$F_{11} = \int_{0.2}^{0.4} \delta(x-0.3)\,\phi_1(x)\,dx = \phi_1(0.3) = 5(0.1) = 1/2$$

Similarly, $F_{12} = 1/2$ since $\phi_2(0.3) = \phi_1(0.3) = 1/2$. Hence, the matrix is:

$$F_2 = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \end{pmatrix}^T$$

NOTE: Many students incorrectly interpreted the boundary conditions to mean from 0 to 1 as usual. Notice that this question is not much more involved than the questions asked; K is tridiagonal with the same elements and F is unchanged, since the load is at $x = 0.3$. No points were deducted for doing this unless K and F were incorrectly computed.